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Self-averaging in the statistical mechanics of some lattice models

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Abstract

We discuss self-averaging of thermodynamic properties in some random lattice models. In particular, we investigate when self-averaging (in an almost sure sense) of the free energy implies self-averaging of the energy and heat capacity, and we discuss the connection between self-averaging in the almost sure sense, and self-averaging in an L^p sense. Under quite general conditions we show that the average of the finite size heat capacity converges to the second derivative of the limiting quenched average free energy. We consider the application of these ideas to the problem of adsorption of a random copolymer at a surface, and to some related systems.

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1. Introduction

Quenched random systems (Brout 1959) have been studied extensively in statistical mechanics. See, for example, Monari *et al* (1999) and papers cited there. One important problem which arises in such systems is the question of self-averaging. Loosely speaking, a property (X) of a system is self-averaging if 'most' realizations of the randomness have the same value of X in the thermodynamic limit. Several different definitions of self-averaging have been used. For instance, the fact that the free energy converges almost everywhere to the limiting quenched average free energy has been proved for spin systems with short-range (van Hemmen and Palmer 1982) and long-range interactions (van Enter and van Hemmen 1983), for self-avoiding walk models of polymer adsorption (Orlandini *et al* 1999) and polymer localization at an interface (Martin *et al* 2000), and for some simplified models of self-interacting polymers (Orlandini *et al* 2000, Janse van Rensburg *et al* 2001). In addition You and Janse van Rensburg (2000) have proved similar results for a model of random branched copolymer adsorption. A definition of self-averaging which has been used largely in numerical studies is as follows. The system is said to be self-averaging with respect to property X if

$$\frac{\langle X_n^2\rangle - \langle X_n\rangle^2}{\langle X_n\rangle^2} \to 0 \tag{1.1}$$

as the size (n) of the system goes to infinity. Here the angular brackets denote averages over the realizations of the quenched randomness of the system and X_n is the value of property X when the system has size n. See, for example, Binder and Young (1986), Aharony and Brooks Harris (1996), and Wiseman and Domany (1998). The relation between these two notions of self-averaging has not received much attention and we make some observations about the connection in section 2.

The work about convergence almost everywhere, referred to above, was confined to the free energy. Corresponding results for the energy are easy to prove (see section 2) but it is not so easy to provide results about self-averaging of the heat capacity. We provide some partial results in this direction in section 3. In section 4 we discuss the application of our results to some problems in the statistical mechanics of random copolymers.

2. Self-averaging of the free energy and energy

We shall be concerned with a system of combinatorial objects in Z^d of size n with an associated colouring $\chi \in Y$ where Y is a probability space. The objects have an energy s which depends on χ. The strength of the interaction is controlled by a parameter β which plays the role of inverse temperature. Let the number of embeddings of these objects be $f_n(s,\chi)$ and let the partition function be

$$Z_n(\beta,\chi) = \sum_s f_n(s,\chi)\,\mathrm{e}^{\beta s}. \tag{2.1}$$

The corresponding free energy is

$$F_n(\beta,\chi) = n^{-1}\log Z_n(\beta,\chi). \tag{2.2}$$

Suppose that the following conditions hold:

C1. $F_n(\beta,\chi) \leqslant M(\beta) < \infty$ for all $\beta < \infty$ and all finite n,

C2. $F_n(\beta,\chi)$ converges almost surely in χ to $F(\beta) < \infty$, for $n \to \infty$, and

C3. There exists a constant $Q < \infty$ and a function $s_{\max}(n) \leqslant Qn$, such that for sufficiently large n, $f_n(s,\chi) = 0$ for all $s > s_{\max}(n)$.

Condition C1 requires that the free energy is finite for any finite size system and for any realization of the disorder, and condition C3 ensures that the energy $\partial F_n(\beta,\chi)/\partial\beta$ is also finite. Condition C2 says that the free energy converges almost surely and we want to investigate conditions under which this (a) implies convergence of the free energy in the sense of (1.1) and (b) implies self-averaging of the energy and heat capacity.

In the above, $F(\beta)$ is the limiting quenched average free energy of the system and sufficient conditions for C2 to hold in some polymer systems are given in Orlandini *et al* (2000).

Proposition 1. *Conditions C1 and C2 imply that $F_n(\beta,\chi)$ converges in $L^p(Y)$ to $F(\beta)$ for all $\beta < \infty$ and for all p such that $1 \leqslant p < \infty$.*

This proposition, which is an immediate consequence of the Lebesgue-dominated convergence theorem, says that almost everywhere convergence and boundedness implies

$$\int_Y |F_n(\beta,\chi) - F(\beta)|^p\,\mathrm{d}\chi \to 0 \tag{2.3}$$

as $n \to \infty$. The case $p = 2$ is equivalent to the notion of self-averaging embodied in (1.1) provided that the denominator in (1.1) has a non-zero limit.

We now turn to the question of self-averaging of the energy of the system. The energy per vertex, $U_n(\beta, \chi)$, is given by

$$U_n(\beta, \chi) = \frac{\partial F_n(\beta, \chi)}{\partial \beta} \tag{2.4}$$

and we are interested in the convergence of $U_n(\beta, \chi)$. We first prove that the free energy is convex.

Lemma 1. *The free energy $F_n(\beta, \chi)$ is a convex function of β for $\beta < \infty$ for any colouring χ.*

Proof. By the Cauchy–Schwartz inequality

$$Z_n\left(\frac{\beta_1 + \beta_2}{2}, \chi\right) \leqslant [Z_n(\beta_1, \chi) Z_n(\beta_2, \chi)]^{1/2} \tag{2.5}$$

so that after taking logarithms and dividing by n we have

$$F_n\left(\frac{\beta_1 + \beta_2}{2}, \chi\right) \leqslant \frac{1}{2}[F_n(\beta_1, \chi) + F_n(\beta_2, \chi)]. \tag{2.6}$$

Since the Z_n are polynomials in e^{β} they are continuous and therefore $F_n(\beta, \chi)$ is a continuous function of β. By a theorem of Hardy *et al* (1934) this implies that $F_n(\beta, \chi)$ is a convex function of β for all χ. □

Remark 1. By condition C2 the sequence $F_n(\beta, \chi)$ converges almost surely to the limit $F(\beta)$, and, since this limit is the limit of a sequence of convex functions, $F(\beta)$ is convex.

Since $F_n(\beta, \chi)$ is the logarithm of a polynomial in e^{β}, where the polynomial is always positive, $F_n(\beta, \chi)$ is smooth for all $\beta < \infty$. It is also a monotone non-decreasing function of β. Its derivative with respect to β, $U_n(\beta, \chi)$, is also a smooth and monotone non-decreasing function of β.

Proposition 2. *$U_n(\beta, \chi)$ converges, almost surely in χ, to $U(\beta) := \frac{\partial F(\beta)}{\partial \beta}$ for almost all β.*

Proof. $F_n(\beta, \chi)$ is a sequence of convex functions, differentiable for all $\beta < \infty$. Therefore, for every β for which the limit of the sequence is differentiable, the sequence of derivatives converges to the derivative of the limit function. The limit of the sequence is $F(\beta)$ which is convex, and so almost everywhere differentiable. Hence the derivative of $F_n(\beta, \chi)$ with respect to β converges, almost surely in χ, to the derivative of $F(\beta)$, for almost all β. □

Proposition 3. *$U_n(\beta, \chi)$ converges to $U(\beta)$ in $L^p(Y)$ for $1 \leqslant p < \infty$.*

Proof. By condition C3, $U_n(\beta, \chi) \leqslant Q$, so it is bounded in $L^p(Y)$. Moreover it converges almost surely in χ to $U(\beta)$ by proposition 2, so the result follows from the Lebesgue-dominated convergence theorem. □

Remark 2. The result of proposition 2 together with the boundedness of U_n (condition C3) implies that

$$\int_Y U_n(\beta, \chi)\, d\chi \to \int_Y U(\beta)\, d\chi = U(\beta). \tag{2.7}$$

Proposition 3 for the case $p = 2$, together with (2.7), implies that

$$\int_Y U_n(\beta, \chi)^2\, d\chi - \left[\int_Y U_n(\beta, \chi)\, d\chi\right]^2 \to 0 \tag{2.8}$$

which establishes (1.1) for the energy of the system.

3. Self-averaging of the heat capacity

In this section we are interested in proving self-averaging of the heat capacity, $C_n(\beta, \chi) = \partial U_n(\beta, \chi)/\partial\beta$. We shall show that, under conditions which are not very restrictive, the average of the heat capacity at finite n, $\int C_n(\beta, \chi)\, \mathrm{d}\chi$, converges to the second derivative of the limiting quenched average free energy, $\partial^2 F(\beta)/\partial\beta^2$. We also find conditions which are sufficient to ensure that the heat capacity self-averages.

Since $U_n(\beta, \chi)$ is not, in general, a convex function of β we cannot apply to $C_n(\beta, \chi)$ the same argument that we used for $U_n(\beta, \chi)$ in proposition 2. Moreover the argument used for $F_n(\beta, \chi)$, in remark 1, cannot be applied to $U_n(\beta, \chi)$ and we cannot say anything *a priori* about the differentiability of $U(\beta)$. Our aim is to prove self-averaging for the heat capacity in a finite interval $B = [\beta_1, \beta_2]$ which can contain points where $U(\beta)$ is not differentiable, corresponding to phase transitions in the system. We shall not be able to give a complete treatment of the problem but we shall be able to say something useful for certain types of phase transition.

Instead of assuming differentiability of $U(\beta)$ in B we make the less restrictive assumption:

C4. $U(\beta) \in AC([\beta_1, \beta_2])$, i.e. the energy belongs to the class of absolutely continuous functions in that interval.

This is equivalent to saying that the derivative, $U'(\beta)$, exists for almost all $\beta \in B$, that $U'(\beta)$ is integrable, and that the fundamental theorem of calculus holds, for all $\beta \in B$ (see for instance, Rudin (1987) or Titchmarsh (1932)). This last property will be crucial in the proof of the next lemma. Condition C4 is very mild and, for instance, does not rule out a first-order phase transition.

Since $C_n(\beta, \chi)$ is the second derivative of a convex function it is always non-negative. Finally we also make the assumption:

C5. The sequence $C_n(\beta, \chi) = \frac{\partial U_n}{\partial \beta}$ is bounded above by a finite function of β, i.e. $C_n(\beta, \chi) \leqslant P(\beta) < \infty$.

Condition C5 requires that the heat capacity is finite so this condition will not hold at a second-order transition at which the heat capacity diverges. However, for such a system this condition would be expected to hold away from the transition. In addition, it would hold in systems in which there is a phase transition signalled by a cusp in the heat capacity.

Lemma 2. *Every weakly convergent subsequence of $C_n(\beta, \chi)$ has the same limit, for fixed $\beta \in B$, and the limit is independent of χ.*

Proof. Since $C_n(\beta, \chi)$ is bounded in $L^p(Y)$, i.e. $\|C_n(\beta, \chi)\|_{L^p(Y)} \leqslant P(\beta) < +\infty$, there exists a subsequence $C_{n_k}(\beta, \chi)$ weakly converging to $t(\beta, \chi)$ in $L^p(Y)$, i.e.

$$\int_Y C_{n_k}(\beta, \chi)\phi(\chi)\, \mathrm{d}\chi \to \int_Y t(\beta, \chi)\phi(\chi)\, \mathrm{d}\chi \qquad \forall \phi \in L^{p'}(Y) \tag{3.1}$$

where $1/p + 1/p' = 1$. This implies that (Brezis 1986)

$$\|t(\beta, \chi)\|_{L^p(Y)} \leqslant \liminf_{k\to\infty} \|C_{n_k}(\beta, \chi)\|_{L^p(Y)}. \tag{3.2}$$

This inequality, together with the bound $C_n(\beta, \chi) \leqslant P(\beta)$ gives $\|t(\beta, \chi)\|_{L^p(Y)} \leqslant P(\beta)$. By the Holder inequality

$$\int_B \left(\int_Y |t(\beta, \chi)\phi(\chi)|\, \mathrm{d}\chi\right) \mathrm{d}\beta \leqslant \int_B \left[\left(\int_Y |t(\beta, \chi)|^p\, \mathrm{d}\chi\right)^{1/p} \left(\int_Y |\phi(\chi)|^{p'} \mathrm{d}\chi\right)^{1/p'}\right] \mathrm{d}\beta$$
$$\leqslant K \int_B P(\beta)\, \mathrm{d}\beta < +\infty \tag{3.3}$$

where K can depend on ϕ. We integrate with respect to β both sides of (3.1) and since $\int_B \left(\int_Y |t(\beta,\chi)\phi(\chi)|\,\mathrm{d}\chi\right)\mathrm{d}\beta < +\infty$, the order of integration can be interchanged by Fubini's theorem. This gives

$$\int_Y \phi(\chi)\left(\int_{\beta_1}^{\beta} C_{n_k}(\gamma,\chi)\,\mathrm{d}\gamma\right)\mathrm{d}\chi \to \int_Y \left(\int_{\beta_1}^{\beta} t(\gamma,\chi)\,\mathrm{d}\gamma\right)\phi(\chi)\,\mathrm{d}\chi. \qquad (3.4)$$

Therefore,

$$\int_Y \phi(\chi)U_{n_k}(\beta,\chi)\,\mathrm{d}\chi - \int_Y \phi(\chi)U_{n_k}(\beta_1,\chi)\,\mathrm{d}\chi \to \int_Y \left(\int_{\beta_1}^{\beta} t(\gamma,\chi)\,\mathrm{d}\gamma\right)\phi(\chi)\,\mathrm{d}\chi \qquad \forall\phi \in L^{p'}(Y). \qquad (3.5)$$

Since, by proposition 3, $U_n(\beta,\chi)$ converges to $U(\beta)$ in $L^p(Y)$ it converges weakly to $U(\beta)$ in $L^p(Y)$. Therefore

$$\int_Y \phi(\chi)[U(\beta) - U(\beta_1)]\,\mathrm{d}\chi = \int_Y \left(\int_{\beta_1}^{\beta} t(\gamma,\chi)\,\mathrm{d}\gamma\right)\phi(\chi)\,\mathrm{d}\chi \qquad \forall\phi \in L^{p'}(Y). \qquad (3.6)$$

We now show that (3.6) implies

$$U(\beta) - U(\beta_1) = \int_{\beta_1}^{\beta} t(\gamma,\chi)\,\mathrm{d}\gamma \qquad \text{for almost all } \chi \in Y. \qquad (3.7)$$

Since $U(\beta) - U(\beta_1) - \int_{\beta_1}^{\beta} t(\gamma,\chi)\,\mathrm{d}\gamma$ belongs to $L^1(Y)$ for each $\beta \in B$, and since ϕ in (3.6) is in $L^{p'}(Y) \supset C_c^{\infty}(Y)$, where $C_c^{\infty}(Y)$ is the set of smooth functions with compact support in Y, then we can apply corollary IV.24 of Brezis (1986), which implies equation (3.7). By the fundamental theorem of calculus, we get

$$\frac{\partial U(\beta)}{\partial\beta} = t(\beta,\chi) \qquad \text{for almost all } \chi \in Y \qquad (3.8)$$

for almost all $\beta \in B$. This implies that $t(\beta,\chi)$ does not depend on χ. If we now choose another subsequence weakly convergent to, say, $v(\beta,\chi)$, we obtain equality (3.8) with, on the right-hand side, $v(\beta,\chi)$ instead of $t(\beta,\chi)$. But this implies that the weak limit is the same, independent of the chosen subsequence, which proves the lemma. We shall call such a limit $C(\beta)$, i.e. $C(\beta) = \partial U/\partial\beta$. □

Remark 3. Consider the sequence $C_n(\beta,\chi)$ which is bounded in $L^p(Y)$. We note that the whole sequence is weakly convergent in $L^p(Y)$ to $C(\beta)$. This is because, whatever subsequence $C_{n_k}(\beta,\chi)$ we extract from $C_n(\beta,\chi)$, the subsequence will be bounded. From each subsequence $C_{n_k}(\beta,\chi)$ it is possible to extract a weakly converging subsubsequence $C_{n_{k_j}}(\beta,\chi)$. By lemma 2 all such subsubsequences converge to the same limit $C(\beta)$. This implies that the whole sequence $C_n(\beta,\chi)$ converges weakly to the same limit $C(\beta)$. This means that

$$\int_Y C_n(\beta,\chi)\phi(\chi)\,\mathrm{d}\chi \to \int_Y C(\beta)\phi(\chi)\,\mathrm{d}\chi \qquad \forall\phi \in L^{p'}(Y). \qquad (3.9)$$

Theorem 1. *For almost all $\beta < \infty$*

$$\int_Y C_n(\beta,\chi)\,\mathrm{d}\chi \to C(\beta) \qquad (3.10)$$

which implies convergence of mean values for the heat capacity.

Proof. The result follows by putting $\phi = 1$ in (3.9). □

We want to have conditions sufficient to establish L^p-convergence of $C_n(\beta,\chi)$ to $C(\beta)$. Motivated by the behaviour of some physical systems we consider the side condition:

C6. Either $C_n(\beta, \chi) \geqslant C(\beta)$ for all sufficiently large n and for almost all $\chi \in Y$, or $C_n(\beta, \chi) \leqslant C(\beta)$ for all sufficiently large n and for almost all $\chi \in Y$.

This condition is more difficult to establish for typical physical systems though we show, in the next section, that it holds in the high-temperature phase for random copolymer adsorption, and for some related problems. Although it has not been established rigorously in the low-temperature phase for these problems, there is numerical evidence that it also holds in these cases.

Theorem 2. *If condition C6 is satisfied then the sequence $C_n(\beta, \chi)$ converges to $C(\beta)$ in $L^p(Y)$, $2 \leqslant p < \infty$, for almost all $\beta \in B$.*

Proof. We give the proof when $C_n(\beta, \chi) \geqslant C(\beta)$. The proof in the second case proceeds in a similar way. Note that

$$\begin{aligned}\int_Y \left(C_n^p(\beta, \chi) - C^p(\beta)\right) \mathrm{d}\chi &= \int_Y \left(C_n^{p-1}(\beta, \chi) - C(\beta)^{p-1}\right) C_n(\beta, \chi)\, \mathrm{d}\chi \\ &\quad + \int_Y (C_n(\beta, \chi) - C(\beta))\, C(\beta)^{p-1}\, \mathrm{d}\chi \\ &\leqslant P(\beta) \int_Y \left(C_n^{p-1}(\beta, \chi) - C(\beta)^{p-1}\right) \mathrm{d}\chi + C(\beta)^{p-1} \int_Y (C_n(\beta, \chi) - C(\beta))\, \mathrm{d}\chi\end{aligned} \tag{3.11}$$

where we have used the fact that heat capacities are non-negative. By (3.10) the second term converges to zero, and

$$\|C_n(\beta, \chi)\|_{L^p(Y)} \to \|C(\beta)\|_{L^p(Y)} \tag{3.12}$$

follows by induction. Since L^p is uniformly convex for $2 \leqslant p < \infty$, (3.9) and (3.12) imply strong convergence in L^p for $2 \leqslant p < \infty$ (Cioranescu and Donato 1999, proposition 1.17). That is,

$$\int_Y |C_n(\beta, \chi) - C(\beta)|^p\, \mathrm{d}\chi \to 0 \tag{3.13}$$

which completes the proof. □

For $p = 2$, this lemma proves the notion of self-averaging contained in (1.1) for the heat capacity $C(\beta)$ a.e. in B. We now consider almost sure convergence of the heat capacity to a limit independent of χ.

Theorem 3. *If condition C6 is satisfied there exists at least one subsequence $C_{n_k}(\beta, \chi)$ which converges almost everywhere to $C(\beta)$, for $\beta \in B$. If there exist other convergent subsequences then all of these converge to the same limit.*

Proof. Since $C_n(\beta, \chi)$ converges in $L^2(Y)$ it converges in $L^1(Y)$ and therefore in measure. Therefore, we can extract a subsequence $C_{n_k}(\beta, \chi)$ which converges almost everywhere (Rudin 1987, chapter 3). The second part of the theorem follows from theorem 2 and the uniqueness of the limit. □

4. Application to some random copolymer problems

We first consider a self-avoiding walk model of random copolymer adsorption (Orlandini *et al* 1999). We consider a sequence of colours $\chi \equiv \{\chi_1, \chi_2, \ldots\}$ where each χ_i is A with probability p and B with probability $1 - p$, and where the χ_i are independent. Consider n-edge self-avoiding walks on the simple hypercubic lattice Z^d with vertices labelled $0, 1, \ldots, n$. We write $(x_i, y_i, \ldots, z_i)$ for the coordinates of vertex i and fix the zeroth vertex at the origin. In addition $z_i \geqslant 0$. We associate the colour χ_i with vertex i for $i = 1, 2, \ldots, n$. We take the parameter s in (2.1) to be the number of vertices labelled A which have z-coordinate equal to zero. That is, the number $f_n(s, \chi)$ in (2.1) is the number of self-avoiding walks, starting at the origin, confined to the half-space $z \geqslant 0$, with colouring χ, having s vertices labelled A in the hyperplane $z = 0$.

The free energy $F_n(\beta, \chi)$ is defined by (2.2) and Orlandini *et al* (1999) established condition C2. Clearly $f_n(s, \chi) = 0$ for all $s > n$, which establishes condition C3. We can take $M(\beta) = \sup_{n>0} \kappa_n(\beta) \leqslant \max[\log(2d), \beta + \log(2d)] < \infty$ for $\beta < \infty$, where $\kappa_n(\beta)$ is the free energy of the corresponding homopolymer (with all vertices labelled A). This establishes condition C1. The results of section 2 imply that:

(1) The free energy converges in $L^p(Y)$, which (taking $p = 2$) implies the validity of (1.1) for the free energy.

(2) The energy $U_n(\beta, \chi)$ converges almost surely in χ and in the L^p sense to $U(\beta) = \partial F/\partial\beta$, for all values of β at which $F(\beta)$ is differentiable. The mean energy converges and this, when we take $p = 2$ in the above, establishes (1.1) for the energy.

For the heat capacity our results show that the mean heat capacity $\int C_n(\beta, \chi)\,\mathrm{d}\chi$ converges to $C(\beta) = \partial^2 F(\beta)/\partial\beta^2$ in any interval $B = [\beta_1, \beta_2]$ in which the energy $U(\beta)$ is differentiable and $C(\beta) < \infty$ (so that condition C4 is satisfied). For this model it is known (Orlandini *et al* 1999) that $F(\beta) = F(0)$ for all $\beta \leqslant 0$ and that there is a critical point $\beta_c > 0$ defined as

$$\beta_c = \sup[\beta | F(\beta) = F(0)]. \tag{4.1}$$

For all $\beta < \beta_c$ (i.e. in the high-temperature or desorbed phase) $C(\beta) = 0$ and $C_n(\beta, \chi) \geqslant 0$, so condition C5 is satisfied and $C_n(\beta, \chi)$ converges to $C(\beta) = 0$ in the L^p sense. Taking $p = 2$ these results imply (1.1) for the heat capacity.

These results also apply to a lattice model of adsorption of branched random copolymers (You and Janse van Rensburg 2000).

Orlandini *et al* (1999) also considered a self-avoiding walk model of random copolymer adsorption in which the χ_i are real numbers chosen (independently) from a given probability distribution such that the mean value of $\chi_i > 0$. For this model all of the above results hold.

Martin *et al* (2000) have considered a self-avoiding walk model of localization of a random copolymer at an interface between two immiscible solvents. The model has two 'energy' variables, α and β and one is interested in the behaviour in the (α, β)-plane. In the delocalized phases the free energy is either independent of α and directly proportional to β or vice versa, so the second derivatives $\partial^2 F/\partial\alpha^2$ and $\partial^2 F/\partial\beta^2$ are both zero. These quantities correspond to heat capacities and condition C6 can be shown to hold. All of the results which we can prove for the 2-colour model of copolymer adsorption can be proved in a similar way for the localization problem.

Another area of interest in random copolymers is the models of self-interacting randomly coloured self-avoiding walks. In the usual version of this model the vertices are coloured A or B independently. If two vertices each coloured A are unit distance apart on the lattice (but not adjacent along the walk) we call this an AA contact, and similarly for BB and AB contacts. The energy of a walk (given a particular sequence of colours) is a linear combination of the numbers of AA, BB and AB contacts. For two particular versions of this model, unfolded walks (Orlandini *et al* 2000) and polygons (Janse van Rensburg *et al* 2001), condition C2 has been shown to hold. Conditions C1 and C3 can be established easily for these models. Our results therefore show that the free energy converges in L^p, the mean energy converges and the energy converges both almost everywhere and in L^p, establishing (1.1) for both the free energy and energy. In any interval in which the energy is absolutely continuous the mean heat capacity converges but we are unable to prove self-averaging of the heat capacity without additional conditions.

5. Discussion

In this paper we have examined conditions under which various thermodynamic functions self-average for quenched random systems. Our main focus has been to prove convergence of these thermodynamic functions both almost everywhere and in L^p. The convergence in L^p is of interest since, for the case $p = 2$, it implies convergence of the variance to zero (as in (1.1)). We have shown that if the free energy converges almost everywhere then it converges in L^p and that the energy then converges both almost everywhere and in L^p, in an interval of β, provided that the free energy is differentiable in this interval.

For the heat capacity the situation is more difficult, and we have been unable to prove anything without additional conditions on the energy. We expect that $U_n(\beta, \chi)$ will be a convex function of β in some ranges of β and a concave function of β in others. If we confine our attention to an interval $\beta_1 \leqslant \beta \leqslant \beta_2$ where $U_n(\beta, \chi)$ is either a concave or a convex function of β, then self-averaging of $C(\beta)$ follows by repeating the argument in lemma 1 and propositions 2 and 3 (with $U_n(\beta, \chi)$ replacing $F_n(\beta, \chi)$). We have derived some results about the heat capacity under the less restrictive conditions that $U(\beta)$ is absolutely continuous in an interval $\beta_1 \leqslant \beta \leqslant \beta_2$ (condition C4), and that the heat capacity is bounded above (condition C5). With these conditions we have shown that the mean heat capacity converges. This is important in numerical studies, where the result is normally assumed in order to estimate the quenched average heat capacity. It remains true even when the heat capacity exhibits cusps or finite jumps (second-order transitions). If we impose the additional condition that the limiting heat capacity is approached from one direction (condition C6) then we have also proved that the heat capacity converges in L^p.

We have considered the application of these results to some problems involving random copolymers, such as random copolymer adsorption and localization, and a self-interacting random copolymer. For the adsorption and localization problems we have shown that the models satisfy conditions C1 to C6 in the high-temperature phases so that the free energy and energy converge almost everywhere and in L^p, and the heat capacity converges in L^p. In the low-temperature phase we can prove results about the free energy and energy but we are unable to say anything about the heat capacity without additional conditions. Our results do not allow us to say anything about the convergence of metric properties such as the radius of gyration.

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